Expansion Around the Mean-Field Solution of the Bak-Sneppen Model

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We study a recently proposed equation for the avalanche distribution in the Bak-Sneppen model. We demonstrate that this equation indirectly relates τ, the exponent for the power law distribution of avalanche sizes, to D, the fractal dimension of an avalanche cluster. We compute this relation numerically and approximate it analytically up to the second order of expansion around the mean field exponents. Our results are consistent with Monte Carlo simulations of Bak-Sneppen model in one and two dimensions.

The Bak-Sneppen (BS) model [1] has become one of the paradigms of Self-Organized Criticality (SOC) [2]. The rules of its dynamics are very simple: the state of the model is completely defined by L^d numbers f_i arranged on a d-dimensional lattice of size L. At every time step the smallest of these numbers and its $2d$ nearest neighbors are replaced with new uncorrelated random numbers, drawn from some distribution $\mathcal{P}(f)$. This "minimalistic" dynamics results in a remarkably rich and interesting behavior. In fact, there exists a whole class of models called *extremal* models [3], which evolve according to similar rules, and share many similar features with the BS model. In all these models the update happens only at the site carrying the global minimum of some variable. The oldest, and perhaps the most widely known of these models is Invasion Percolation [4]. The BS model, being the simplest and the most analytically treatable extremal model, occupies the place of an "Ising model" in this class.

The self-organized critical nature of the BS model (as well as of other extremal models) is revealed in its ability to naturally evolve towards a stationary state where almost all the variables f_i are above a critical threshold f_c. The dynamics in the stationary state is characterized by scale-free bursts of activity or *avalanches*, which form a hierarchical structure [1,5] of sub-avalanches within bigger avalanches. The introduction of an auxiliary parameter f [3] allows one to describe the system within the paradigms of standard critical phenomena. Indeed the distribution $P(s, f)$ of avalanche sizes s close to f_c, has the same qualitative behavior of the cluster distribution of percolation [6] above and below the critical threshold p_c: For $f < f_c$, $P(s, f)$ has a finite cut-off, reminiscent of an undercritical system. As $f \to f_c$ the cut-off diverges and a scale-free distribution $P(s, f_c) \sim s^{-\tau}$ emerges. In the overcritical regime $f > f_c$ there is a non-zero probability to start an infinite avalanche, but all finite avalanches again have a finite cut-off. Scaling arguments [3] allow one to derive all critical exponents of a general extremal model in terms of only two independent ones, say τ and D – the fractal dimension of the avalanche cluster.

In order to compute these two remaining exponents one has to resort to methods which go beyond scaling arguments. Apart from the solution of the mean field case [7], and a real space renormalization group approach [8] for $d = 1$, a systematic theory to compute the BS exponents is still lacking. A promising step in this direction was recently taken by one of us [9] with the introduction of an *exact* equation for the avalanche distribution. Hereafter we will refer to this equation as the Avalanche Hierarchy Equation (AHE). It was shown that inside the AHE is hidden an infinite series of equations, relating different moments of the avalanche size distribution.

In this letter we demonstrate that, as it was conjectured in [9], the AHE indirectly relates these two exponents, thus reducing the number of independent critical exponents in the BS model to just one. Contrary to simple rational relations based on scaling arguments [3], this exponent relation is highly non-trivial. First we display the numerical solution of the AHE. Then we perform a perturbative "ϵ"-expansion around the mean field solution, up to the second order in ϵ. The numerical solution of AHE is in agreement with the MC simulations in $d = 1, 2$ [3,10] and is well approximated by the results of the ϵ expansion up to second order. They constitute a significant step forward towards the full solution of the BS model. However, the unusual type of the expansion around the mean-field exponents leaves open the question of the upper critical dimension d_c in the model. It also does not answer the question about the geometrical, fractal properties of the avalanche cluster. Instead, given an avalanche fractal dimension D it enables one to derive the power τ of the avalanche distribution. Similarly, in ordinary percolation the power of cluster distribution τ is related to the cluster's fractal dimension D via a hyperscaling relation $\tau = 1 + d/D$ [6].

Following ref. [9], let us consider the exponential distribution $\mathcal{P}(f) = e^{-f}$, $f > 0$. This simplifies the expressions without loss of generality [11]. To define avalanches one records the *signal* of the model, i.e. the value of the global minimal number $f_{min}(t)$ as a function of time t.

Then for every value of an auxiliary parameter f, an f-avalanche of size (temporal duration) s is defined as a sequence of $s-1$ successive events, when $f_{min}(t) < f$, confined between two events, when $f_{min}(t) > f$. In other words, the events, when $f_{min}(t) > f$, divide the time axis into a series of avalanches, following one another. The AHE is an equation for the probability distribution $P(s,f)$ of f-avalanche sizes s, and it reads [9]: $\partial_f P(s,f) = \sum_{s_1=1}^{s-1} R^d(s_1)P(s_1,f)P(s-s_1,f) - R^d(s)P(s,f)$. Here $R^d(s)$ is the average number of distinct sites updated at least once during an avalanche of size s. The equation describes how the sequence of avalanches changes when the value of f is raised by an infinitesimal amount df. The first term on the RHS describes the gain of avalanches of size s due to merging of two consecutive avalanches of size s_1 and $s-s_1$. Such merging occurs when the value of $f_{\min}(t)$, terminating the first avalanche, happens to be in the interval $[f, f+df]$. The factor $R^d(s_1)$ comes from the fact that when avalanches merge, the active site starting the second avalanche must be one of the sites updated in the first avalanche. The bigger is the region covered by an avalanche, the better are its chances to merge with the one directly following it. The second term in the RHS is the "loss term" due to avalanches of size s merging to form a larger avalanche.

To proceed further one needs to introduce the scaling ansatz $R^d(s) \sim s^\mu$ for the number of updated sites. This power law relation is a consequence of the *spatio-temporal* fractal structure of the avalanches in the BS model [12]. The exponent $\mu(d)$, which is an independent "input" variable in the AHE, depends on the dimensionality of the model and the fractal structure of the avalanche. Physically, the exponent μ relates the volume of the spatial projection of an avalanche cluster to its temporal duration s. If this spatial projection is a dense object with a fractal dimension equal to the dimension of space d, μ is given by d/D. In this expression D is the fractal dimension of the avalanche [3] defined through $s = R^D$. This is known to be true in $d=1$, where the connected nature of an avalanche cluster ensures the compactness (absence of holes) of its projection. In $d=2$ the projection of the avalanche can have holes, but still it was numerically found to be dense (i.e have a fractal dimension d) [3]. It is clear that, as the dimensionality of space is increased, the exponent μ should approach 1, since multiple updates of the same site become less and less likely and the volume of the projection should be closer and closer to the total volume $(2d+1)s$ of the avalanche itself. The "hyperscaling" relation $\mu = d/D$ will be clearly violated for $d > d_c$, where one has $D=4$, $\mu=1$ [3,13]. From this it follows that $d_c \geq 4$ for the BS model.

The introduction of the "phenomenological" exponent μ closes the AHE, which then reads

$$\partial_f P(s,f) = \sum_{s_1=1}^{s-1} s_1^\mu P(s_1,f)P(s-s_1,f) - s^\mu P(s,f). \quad (1)$$

The solution of equation (1) exhibits a power law behavior $P(s,f) \sim s^{-\tau}$ when f is at its critical value f_c. Close to f_c it takes a scaling form

$$P(s,f) = s^{-\tau} F(s^\sigma \Delta f), \quad (2)$$

where $\Delta f = f_c - f$. The exponents τ, σ, and μ are related through $\tau = 1 + \mu - \sigma$ [3]. Perhaps a more familiar form of this exponent relation involves the correlation length exponent $\nu = 1/\sigma D$. The relation then becomes $\tau = 1 + (d - 1/\nu)/D$. It has been conjectured [9] that equation (1) also indirectly relates the exponents μ and τ. In order to check this conjecture we numerically integrated Eq. (1) forward in f with the initial condition $P(s,0) = \delta_{s,1}$ for several values of μ [14]. In order to locate the critical point $f_c(\mu)$, a least square fit of $\log P(s,f)$ vs $\log s$ was performed runtime for each value of f. The value $\chi^2(f)$ of the sum of the squared distances from the fit drops almost to zero in a very narrow region (see Fig. 1), which then allows for a very precise estimate of $f_c(\mu)$ and $\tau(\mu)$. The results for the latter are shown in figure 2 ($\Box$). These results are in a perfect agreement with Monte Carlo estimates of τ in one and two dimensions [3,10]. In $d=1$, $\mu = 1/D = 0.411(2)$ and $\tau = 1.07(1)$, while the results of numerical integration of (1) give $\tau(0.4) = 1.058$. In $d=2$ the Monte Carlo results $\mu = 2/D = 0.685(5)$, $\tau = 1.245(10)$ are also consistent with our relation giving $\tau(0.7) = 1.238$. This confirms that Eq. (1) indeed contains a *novel* non-trivial relation between τ and μ.

In order to address this relation analytically, let us take the Laplace transform of Eq.(1) [9]. The AHE, with $p(\alpha,f) \equiv \sum_{s=1}^{\infty} P(s,f)e^{-\alpha s}$, reads

$$\partial_f \ln[1 - p(\alpha,f)] = \sum_{s=1}^{\infty} P(s,f) s^\mu e^{-\alpha s}. \quad (3)$$

$p(\alpha,f)$ has the scaling form given by:

$$p(\alpha,f) = 1 - \alpha^{\tau-1} h\left(\Delta f/\alpha^\sigma\right). \quad (4)$$

This scaling form follows from Eq.(2) and the scaling functions are related through $h(x) = \int_0^\infty [F(0) - F(xy^\sigma)e^{-y}]y^{-\tau}dy$.

The scaling function $h(x)$ (as well as $F(x)$) is analytic at $x=0$. Its large $|x|$ asymptotics is determined by the fact that at any $\Delta f \neq 0$, $p(\alpha,f)$ is analytic in α, since $p(\alpha,f) = 1 - P_\infty(f) + \langle s\rangle_f \alpha + \langle s^2\rangle_f \alpha^2 + \ldots$, and all the moments of $P(s,f)$ are finite except at the critical point. Here $P_\infty(f)$ is the probability to start an infinite avalanche, and $P_\infty(f) = 0$ for $f < f_c$. Matching the expected behavior of $p(\alpha,f)$ to its scaling form one gets:

$$h(\pm|x|) = |x|^{(\mu-\sigma)/\sigma} \sum_{k=0}^{\infty} b_k^\pm |x|^{-k/\sigma}. \quad (5)$$

In other words, $h(x)|x|^{(\sigma-\mu)/\sigma}$ for $|x| \gg 1$, must be an analytic function of $|x|^{-1/\sigma}$. The coefficients $b_k^\pm$ for $k>0$ are related to the amplitudes of the diverging moments through $\langle s^k \rangle_f = (-1)^{k+1} b_k^\pm |f_c - f|^{-(k+\sigma-\mu)/\sigma}$, in the under-critical (b_k^+, $f<f_c$) and in the over-critical regime (b_k^-, $f>f_c$). As we will see later, it is the condition that $h(x)$ has the desired asymptotic form (5), which fixes the value of σ for a given μ.

With the scaling ansatz (4), the LHS of Eq.(3) becomes $-\alpha^{-\sigma}h'(x)/h(x)$, where $x = \Delta f/\alpha^\sigma$. The RHS needs some more work: using $s^\mu e^{-\alpha s} = -\partial_\alpha s^{\mu-1}e^{-\alpha s}$ and the identity $s^{\mu-1} = \int_0^\infty t^{-\mu}e^{-st}dt/\Gamma(1-\mu)$, one can express the RHS of Eq. (3) in terms of an integral involving $p(\alpha, f)$. Matching the powers of α in the resulting equation for $h(x)$ one gets once more the well known exponent relation:

$$\tau = 1 + \mu - \sigma. \tag{6}$$

After eliminating τ, equation (3) finally reads:

$$h'(x) = \frac{h(x)}{\Gamma(1-\mu)} \int_0^1 dz \frac{xzh'(xz) - \frac{\mu-\sigma}{\sigma}h(xz)}{(1-z^{1/\sigma})^\mu} \tag{7}$$

We were not able to solve equation (7) and find the exact relation $\sigma(\mu)$. However, we can explicitly solve AHE for $\mu = 1$. This corresponds to the mean field version of the BS model, which has been studied in detail [7]. We will rederive their results using our approach. Our strategy will then be to perform a systematic $\epsilon-$expansion around the mean-field solution, where $\epsilon = 1-\mu$. This clearly differs from the standard $\varepsilon = d - d_c$ expansion (note that the upper critical dimension d_c for the BS model is still an open issue), since the dimensionality of the system does not enter directly into our discussion.

As $\mu \to 1$, the integral in Eq. (7) diverges for $z \simeq 1$, but so does $\Gamma(1-\mu)$. This implies that the factor $[\sigma\Gamma(1-\mu)(1-z^{1/\sigma})^\mu]^{-1}$ behaves as a $\delta(z-1)$ function. For $\mu = 1$ Eq. (7) reduces to

$$h'(x) = \sigma x h(x) h'(x) - (1-\sigma)h^2(x). \tag{8}$$

Its solution reads $h(x)[1 - xh(x)]^{\sigma-1} = a_0$, with a_0 an integration constant. Eq. (5) implies a large x behavior $h(x) \simeq x^{-1}(C + Dx^{-1/\sigma} + \ldots)$, compatible with the solution of Eq.(8) only if $C = 1$ and $\sigma = 1 - \sigma = 1/2$. For $a_0 = 1$ one recovers a mean field solution [7]:

$$h^{(0)}(x) = \frac{\sqrt{4+x^2} - x}{2}. \tag{9}$$

The above derivation demonstrates that the knowledge of the whole scaling function $h(x)$ is necessary in order to find the exponent σ.

To proceed beyond the mean field case we perform a $1-\mu \equiv \epsilon-$expansion of Eq. (7) around the $\mu = 1$ solution (9). We put $\sigma = \frac{1}{2} + c\epsilon$, where c is to be determined later. It is convenient to change variables to $z = h^{(0)}(x)$ and to set $h(x) = z[1 + \epsilon\phi(z) + O(\epsilon^2)]$. Keeping only terms linear in ϵ one gets an equation for ϕ:

$$\frac{1}{2}z(1+z^2)\partial_z\phi = \phi - 1 - \psi(1/2) + (1+2c)z^2 + 2\ln\frac{1+z^2}{2z},$$

where $\psi(x)$ is the logarithmic derivative of $\Gamma(x)$. This equation has to be solved with the boundary condition $\phi(z=1) = 0$ (i.e. $h(0) = 1$). After some algebra one gets:

$$\phi(z) = A\frac{1-z^2}{1+z^2} - 2\ln\frac{1+z^2}{2} + \frac{2+(2-4c)z^2}{1+z^2}\ln z \tag{10}$$

where $A = 2 + \psi(1/2) \cong 0.03649\ldots$. The value of c is set by the requirement that $\phi(z)$ must give rise to the desired asymptotic behavior of $h(x)$. The singular behavior at $z \simeq 1/x \to 0$ must be matched to the asymptotics of $h(x)$ for $x \to \infty$. To order ϵ, Eq. (5) requires that $h(x) = x^{-1-2\epsilon}f(x^{-2+4c\epsilon})$, where $f(y)$ is analytic at $y = 0$ [to order ϵ^0, $f(y) = (\sqrt{1+4y} - 1)/2$]. Expanding this relation to order ϵ one finds that the singular part of $\phi(z)$ must have *exactly* the form $2[1 + 2cz^2/(1+z^2)]\ln z$. The only value of c which matches this requirement to the last term in the RHS of Eq. (10) is $c = 0$. Note that the whole asymptotic behavior, and not just its leading part, is necessary to determine c.

This concludes the first order of the expansion in ϵ. We have found that in the first order in ϵ the critical exponent σ did not change. The exponent relation (6) then gives $\tau = 3/2 - \epsilon$. Finally, the analytic form of the scaling function $h(x)$, containing all information about the amplitudes of avalanche moments, is given by

$$h(x) = \frac{\sqrt{4+x^2} - x}{2}\left[1 + \frac{x^2}{4}\right]^{-\epsilon}\left[1 + \frac{\epsilon A x}{\sqrt{4+x^2}}\right] + O(\epsilon^2).$$

The extension of this procedure to higher orders in ϵ is straightforward, even though it involves much heavier algebra. Skipping the details [15], up to second order in ϵ we find

$$\begin{aligned}\sigma &= \frac{1}{2} - \frac{4}{3}(\gamma + \ln 2 - 1)\epsilon^2 + O(\epsilon^3) \\ &\simeq 0.5 - 0.3605\epsilon^2 + O(\epsilon^3); \qquad (11)\\ \tau &\simeq 1.5 - \epsilon + 0.3605\epsilon^2 + O(\epsilon^3). \qquad (12)\end{aligned}$$

Here $\gamma \simeq 0.5772$ is the Euler's constant. The explicit expression for $h(x)$ at this order is not particularly illuminating, so we refrain to display it here. As seen in Fig. 2, the expansion up to the first two orders is in excellent agreement with numerical data down to $\mu \approx 0.6$ ($\epsilon \approx 0.4$).

On the other side, Fig. 2 seems to suggest a singular behavior of $\tau(\mu)$ as $\mu \to 0$. The specialty of $\mu = 0$ can be understood by observing that in this case $P(s, f)$ does not obey scaling. Indeed, $\mu = 0$ corresponds to a trivial model with only one constantly updated site, which can

be considered as a 0-dimensional lattice. The probability of f-avalanches of size s is trivially derived from the probability $P(f_{\min}(t) < f) = 1 - e^{-f}$ that the signal is below f: $P(s,f) = e^{-f}(1-e^{-f})^{s-1}$. This is indeed the solution of Eq. (1) with $s^{\mu} = 1$. There is no phase transition (numerically we found $f_c(\mu) \sim 1/\mu \to \infty$ as $\mu \to 0$) and the avalanche distribution always has an exponential cutoff. This suggests that $d = 0$ can be interpreted as the lower critical dimension for the BS model (note that $P(s,f)$ in the $d = 0$ BS model is very similar to the cluster size distribution in the $d = 1$ percolation [6]).

In conclusion, we have shown that the avalanche hierarchy equation introduced by one of us in [9] yields a new relation between exponents in the Bak-Sneppen model, thus reducing the number of independent exponents to just one. This relation expresses τ, the power law exponent in the avalanche probability distribution, in terms of D, the mass dimension of an avalanche cluster. We were able to perform a systematic expansion of this relation around the mean field exponents, carried to the second order in this work. The success of this approach suggests that a complete $\varepsilon = d - d_c$ expansion for the BS model could be possible. The accomplishment of this task, however, calls for a systematic study of the BS model in high dimensions and for identification of the upper critical dimensionality d_c.

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[11] The BS model is "insensitive" to the shape of random number distribution because only the *order* among variables f_i matters in the dynamics. With the rescaling map $f(f') = -\ln \int_{f'}^{\infty} P(f')df'$, the BS dynamics for a general $P(f')$ is turned into that with the exponential distribution.
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[13] Indeed a cluster in the BS model above the upper critical dimension has the same fractal dimension $D = 4$ as the cluster in the mean-field ordinary or directed percolation [6], as well as any other mean-field branching process.
[14] Eq. (1) is a system of coupled differential equations one for each value of s. The equation for $P(s,f)$ couples only to $P(s',f)$ for $s' < s$ and therefore the system of equations can be solved for $s \leq s_{\max}$ without approximations. We integrated with $s_{\max} = 512, 1024$ and 2048. The exponent τ slightly increases with $s_{\max}$ thus yielding a systematic error which we estimate to be less than 1%.
[15] P. De Los Rios, M. Marsili and S. Maslov, to be published.

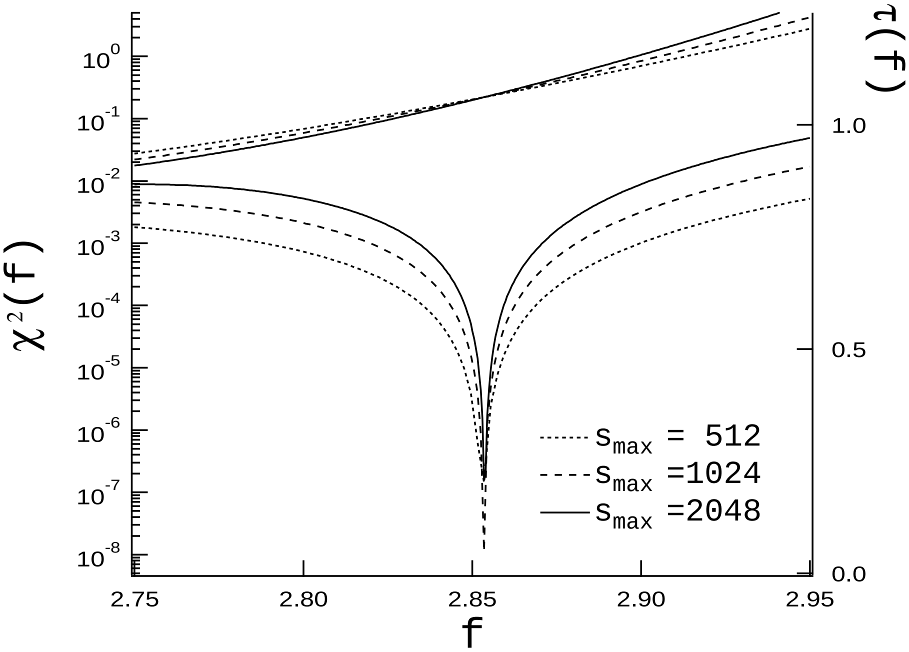

FIG. 1. The plot of χ^2 and τ vs f for $\mu = 0.411$ and $s_{\max} = 521,\ 1024$ and 2048.

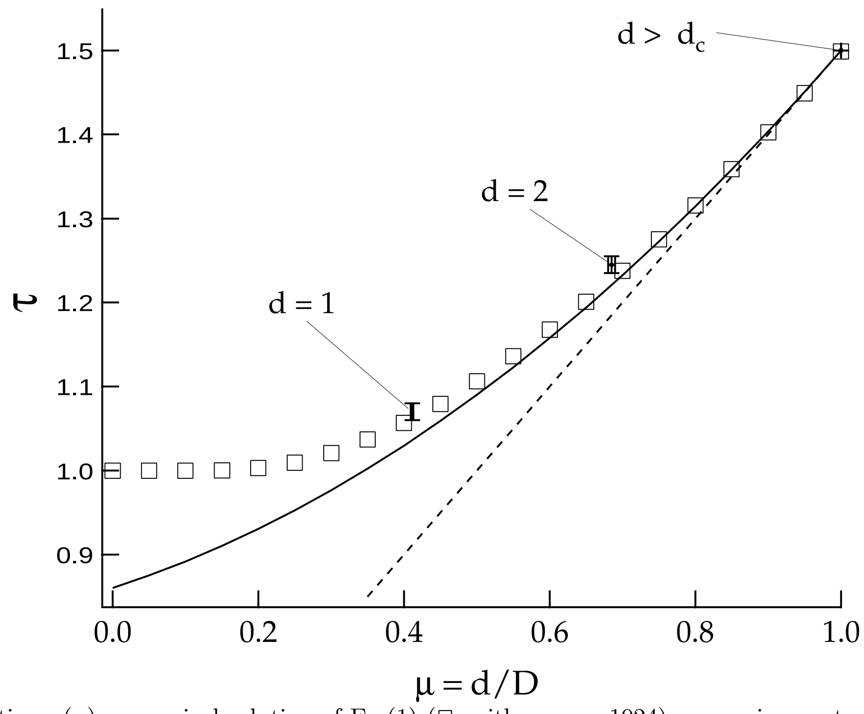

FIG. 2. The relation $\tau(\mu)$: numerical solution of Eq.(1) ($\square$, with $s_{\max} = 1024$), expansion up to order $1 - \mu$ (dashed line) and $(1 - \mu)^2$ (full line). The results of the Monte Carlo numerical simulations in $d = 1$, 2 [3] and the mean field result [7] are also shown.